Mean field theory and coherent structures for vortex dynamics on the plane

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Abstract

We present a new derivation of the Onsager-Joyce-Montgomery (OJM) equilibrium statistical theory for point vortices on the plane, using the Bogoliubov-Feynman inequality for the free energy, Gibbs entropy function and Landau's approximation. This formulation links the heuristic OJM theory to the modern variational mean field theories. Landau's approximation is the physical counterpart of a large deviation result, which states that the maximum entropy state does not only have maximal probability measure but overwhelmingly large measure relative to other macrostates.

1 Introduction

The so-called mean field equation of the Onsager-Joyce-Montgomery (OJM) theory[1,2,3] for the equilibrium vorticity distributions of a two-dimensional inviscid and incompressible fluid (and guiding center line charge model in plasmas) were derived more than twenty years ago but the volume of research concerning these equations continues unabated. Recently, the mean field thermodynamic limit for the planar vortex model has been shown to exist by Caglioti et al [4], and independently by Kiessling[5], and the existence of negative temperature states in this model was demonstrated by Eyink and Spohn [6]. Furthermore, these authors established the maximum entropy principle[7] for this theory. Onsager[1] predicted that the negative temperature equilibrium states exhibit a clustering of the point vortices into coherent structures. Exact solutions of the OJM mean field equations obtained by Chen et al [8] confirm the existence of such coherent states in inviscid turbulent flows.

A OJM type equilibrium statistical theory for nearly parallel thin vortex filaments was very recently put forth by Lions and Majda[9]. They established the mean field thermodynamic limit and the maximum entropy principle for their problem. The mean field equation that Lions and Majda obtained, is radically different from other mean field theories for vortex dynamics in the sense that their equations are time-dependent. Nonetheless, in a special singular limit of perfectly parallel vortex filaments, their mean field equation reduces to the sinh-Poisson mean field equation of the planar point vortex problem.

Concerning the relationship between the OJM and other statistical theories, Majda and Holen[10] showed that both the OJM theory and Kraichnan's truncated spectral theory[11] are statistically sharp with respect to the recent infinite constraints theory of Miller[12,13] and Robert[14], i.e., the few constraints and infinite constraints theory agree for low energies. Chorin[15] has earlier indicated that a few constraints is sufficient for a reasonable theory of statistical equilibrium in these models. He also demonstrated numerically that the Miller-Robert theory is valid only for moderate temperatures and has no Kosterlitz-Thouless phase-transition except at zero temperature[16]. Turkington and Whittaker[17] gave a numerical algorithm for the Miller-Robert theory.

Recently, Turkington[18] critiqued the Miller-Robert theory and presented a few constraints equilibrium statistical theory for 2-D coherent structures, where the equality constraints of the Miller-Robert theory are replaced by inequalities. Turkington[18] argued that the finite ultra-violet cutoff implicit in the Miller-Robert theory does not reflect the true inviscid vortex dynamics as represented in the two dimensional Euler equations. In their preprint[19], Boucher, Ellis, and Turkington presented rigorous large deviation results and proved the existence of the mean field thermodynamic limit for Turkington's theory.

Yet, several issues remain unresolved —in spite of the above recent works, it is still not completely clear where the OJM theory stands in relation to other equilibrium statistical theories of two dimensional turbulence[11,12,13,14] especially with regard to their relative efficacy in modelling real physical problems in fluid mechanics. Moreover, it is still not clear how the OJM theory is related to the physicists' standard variational formulations of mean field theory such as in Chapter 6 and 7 of the text[20]. It is the main aim of this paper to derive equation (1) in a new way, thereby demonstrating that the OJM theory is indeed a mean field theory in the sense of Weiss, Peierls and Feynman[20] . Specifically, we will prove that equation (1) is the limit, as the number of particles $N \to \infty$, of mean field equations that are derived using the Bogoliubov-Feyman inequality[21] (2) , the Gibbs entropy function and Landau's approximation[20]. Previous derivations of the OJM theory[1,2,3] are based on Boltzmann's entropy function instead of Gibbs' entropy function. We note here the significant point that our result holds for all initial vorticity distributions $q_o(x)$.

A OJM-type theory for point vortex systems on a rotating sphere[22] has recently been obtained by Lim[23], and the corresponding mean field thermodynamic limit was established by methods similar to those of Kiessling[5].

2   Mean field theory

All the above statistical theories for two dimensional turbulence in inviscid fluids have a common thread in the maximum entropy principle of information theory[7]. They lead to nonlinear elliptic equations, the so-called mean field equations which have the generic form

$$\Delta \Psi = F(\Psi, \gamma)$$

for the stream function Ψ of the mean field locally averaged vorticity distributions $\bar{q}(x) = -\Delta\Psi$, and some parameters γ. Indeed it can be shown that $\bar{q}(x)$ are exact stationary solutions of the Euler equations. The mean field equation of the OJM theory for a single component vortex gas is the nonlinear elliptic equation

$$\Delta\Psi = ke^{-\beta\Psi}. \tag{1}$$

Heuristically, this equation has the correct form for a mean field theory in the sense that the precise interactions between particles has been replaced by an approximate energy expression where the particles interact with some *mean* field.

The main tool for the work in this section is the Bogoliubov-Feynman inequality[20] for the free energy in the form proven by Feynman[21]:

$$F \leq F_0 + \langle H_1 \rangle_0 \equiv F_{var}, \tag{2}$$

where F_0 is the free energy based on the approximate Hamiltonian H_0, where $H = H_0 + H_1$, and the second term on the right is the thermal average of the remainder H_1 in the canonical measure based on H_0. In other words,

$$\langle \cdot \rangle_0 \equiv \frac{\int_C dV \; (\cdot) \; \exp\{-\beta H_0\}}{Z_0}, \tag{3}$$

where the partition function $Z_0 = \int_C dV \exp\{-\beta H_0\}$ is based on the approximate Hamiltonian H_0. In the standard mean field theory[20], one choose H_0 so that Z_0 can be evaluated easily, and yet it must have certain basic physical properties to make the theory physically relevant. For negative temperatures, a derivation similar to Feynman's yields

$$F \geq F_0 + \langle H_1 \rangle_0 \equiv F_{var}^-. \tag{4}$$

For the OJM theory of point vortices in the plane where the vortices are identical and have vorticity or charge λ, we shall choose the coarse-grained approximate Hamiltonian based on the division of the physical domain $\Omega \subseteq R^2$ of area A into M equal boxes,

$$H_0 = -\frac{1}{2}\sum_{i=1}^{M}\sum_{j\neq i}^{M} n_i n_j \lambda^2 \log |\vec{x}_i^0 - \vec{x}_j^0|.$$

Here n_i denotes the number of vortices in box B_i which has area h^2, and $\vec{x}_i^0$ is the location of the center of B_i. Furthermore, the total number of particles

4

is N, i.e.,
$$\sum_{i=1}^{M} n_i = N.$$

Thus, $\vec{x}_i^0$ are no longer dependent on time but depend instead on the lattice which is implicit in the coarse-grained Hamiltonian H_0. Since the full Hamiltonian of the point vortex model is
$$H = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \lambda^2 \log |\vec{x}_i - \vec{x}_j|,$$

the remainder H_1 is given by
$$H_1 = H - H_0$$

$$= -\frac{1}{2} \sum_{i=1}^{M} \left(\sum_{j=1}^{n_i} \sum_{k \neq j}^{n_i} \lambda^2 \log |\vec{x}_j - \vec{x}_k| \right)$$

$$-\frac{1}{2} \sum_{i=1}^{M} \sum_{i' \neq 1}^{M} \left[\left(\sum_{j=1}^{n_i} \sum_{k=1}^{n_{i'}} \lambda^2 \log |\vec{x}_j - \vec{x}_k| \right) - n_i n_{i'} \lambda^2 \log |\vec{x}_i^0 - \vec{x}_{i'}^0| \right]$$

$$= -\frac{1}{2} \sum_{i=1}^{M} \left(\sum_{j=1}^{n_i} \sum_{k \neq j}^{n_i} \lambda^2 \log |\vec{x}_j' - \vec{x}_k'| \right)$$

$$-\frac{1}{2} \sum_{i=1}^{M} \sum_{i' \neq 1}^{M} \left[\left(\sum_{j=1}^{n_i} \sum_{k=1}^{n_{i'}} \lambda^2 \log |(\vec{x}_i^0 - \vec{x}_{i'}^0) + (\vec{x}_j' - \vec{x}_k')| \right) - n_i n_{i'} \lambda^2 \log |\vec{x}_i^0 - \vec{x}_{i'}^0| \right],$$

where we have used
$$\vec{x}_j - \vec{x}_k = (\vec{x}_i^0 - \vec{x}_i^0) + (\vec{x}_j' - \vec{x}_k') = \vec{x}_j' - \vec{x}_k',$$

in the first sum, and
$$\vec{x}_j - \vec{x}_k = (\vec{x}_i^0 - \vec{x}_{i'}^0) + (\vec{x}_j' - \vec{x}_k')$$

in the second sum. In other words the primed vector $\vec{x}_j'$ denotes the difference $\vec{x}_j - \vec{x}_i^0$, which represents the vector from the center of B_i to vortex j in it.

From
$$
\begin{aligned}
(\vec{x}_i^0 - \vec{x}_{i'}^0) + (\vec{x}_j' - \vec{x}_k') &= (\vec{x}_i^0 - \vec{x}_{i'}^0) + (\vec{x}_j' - \vec{x}_k') \\
&= (\vec{x}_i^0 - \vec{x}_{i'}^0) \left(1 + \frac{(\vec{x}_j' - \vec{x}_k')}{(\vec{x}_i^0 - \vec{x}_{i'}^0)} \right),
\end{aligned}
$$

one derives

$$\log |(\vec{x}_i^0 - \vec{x}_{i'}^0) + (\vec{x}_j' - \vec{x}_k')| = \log |(\vec{x}_i^0 - \vec{x}_{i'}^0)| + \log \left(1 + \frac{|(\vec{x}_j' - \vec{x}_k')|}{|(\vec{x}_i^0 - \vec{x}_{i'}^0)|} \right)$$

$$= \log |(\vec{x}_i^0 - \vec{x}_{i'}^0)| + \frac{|(\vec{x}_j' - \vec{x}_k')|}{|(\vec{x}_i^0 - \vec{x}_{i'}^0)|} + O \left(\frac{|(\vec{x}_j' - \vec{x}_k')|}{|(\vec{x}_i^0 - \vec{x}_{i'}^0)|} \right)^2 ,$$

using the complex form of the logarithm, a step which we omit here for the sake of brevity. Substituting this last inequality back into H_1 and keeping only terms of order $|(\vec{x}_j' - \vec{x}_k')|/|(\vec{x}_i^0 - \vec{x}_{i'}^0)|$, we get

$$H_1 = -\frac{1}{2} \sum_{i=1}^{M} \left(\sum_{j=1}^{n_i} \sum_{k \neq j}^{n_i} \lambda^2 \log |\vec{x}_j' - \vec{x}_k'| \right) - \frac{1}{2} \sum_{i=1}^{M} \sum_{i' \neq 1}^{M} \left(\sum_{j=1}^{n_i} \sum_{k=1}^{n_{i'}} \lambda^2 \frac{|(\vec{x}_j' - \vec{x}_k')|}{|(\vec{x}_i^0 - \vec{x}_{i'}^0)|} \right) ,$$

where the first group of terms represent the intra-box interaction energy, and the second group represents $O(h)$ terms in the inter-box interaction energy.

To begin, we compute the partition function Z_0 which is based on the approximate Hamiltonian H_0, as follows

$$Z_0 = \sum_s W(s) h^{2N} \exp\left(-\beta H_0\right)$$

$$= \sum_s W(s) h^{2N} \exp\left(\frac{\beta}{2} \sum_{i=1}^{M} \sum_{j \neq i}^{M} n_i n_j \lambda^2 \log |\vec{x}_i^0 - \vec{x}_j^0| \right) ,$$

where the summation is taken over all coarse-grained state (or macro-state) $s = (n_1, ..., n_M)$ such that $\sum_{i=1}^{M} n_i = N$, and the degeneracy of the macro-state s is given by

$$W(s) \equiv \frac{N!}{n_1! ... n_M!}.$$

The probability distribution $P_0(s)$ for a macrostate s, is now defined by:

$$P_0(s) = \int_{D^N(s)} \frac{\exp\left(-\beta H_0\right)}{Z_0} d^N A \tag{5}$$

$$= \frac{W(s) h^{2N} \exp\left(-\beta H_0\right)}{Z_0},$$

where $D^N(s)$ denotes the part of phase space D^N which is occupied by the microstates associated with s. Finally, we compute the free energy F_0 as

follows:

$$F_0 = -\frac{1}{\beta}\log Z_0$$

$$= -\frac{1}{\beta}\left[N\log h^2 + \log\left\{\sum_s W(s)\exp\left(\frac{\beta}{2}\sum_{i=1}^M\sum_{j\neq i}^M n_i n_j \lambda^2 \log|\vec{x}_i^0 - \vec{x}_j^0|\right)\right\}\right].$$

In the next step in this procedure, we will compute $\langle H_1\rangle_0$ up to $O(h):-$

$$\langle H_1\rangle_0 = \int_{D^N} H_1(\vec{x})\, P_0(\vec{x})\, d\vec{x}$$

$$= \int_{D^N} H_1(\vec{x})\frac{\exp\left(-\beta H_0(\vec{x})\right)}{Z_0}d\vec{x}_1...d\vec{x}_N$$

$$= \int_{D^N}\left[\begin{array}{c}-\frac{1}{2}\sum_{i=1}^M\left(\sum_{j=1}^{n_i}\sum_{k\neq j}^{n_i}\lambda^2\log|\vec{x}_j' - \vec{x}_k'|\right)\\ -\frac{1}{2}\sum_{i=1}^M\sum_{i'\neq i}^M\left(\sum_{j=1}^{n_i}\sum_{k=1}^{n_{i'}}\lambda^2\frac{|(\vec{x}_j' - \vec{x}_k')|}{|(\vec{x}_i^0 - \vec{x}_{i'}^0)|}\right)\end{array}\right]\frac{\exp\left(-\beta H_0(\vec{x})\right)}{Z_0}d\vec{x}_1...d\vec{x}_N$$

$$= \int_{D^N}\left[-\frac{1}{2}\sum_{i=1}^M\left(\sum_{j=1}^{n_i}\sum_{k\neq j}^{n_i}\lambda^2\log|\vec{x}_j' - \vec{x}_k'|\right)\right]\frac{\exp\left(-\beta H_0(\vec{x})\right)}{Z_0}d\vec{x}_1...d\vec{x}_N\ + O(h).$$

where $\vec{x}\equiv(\vec{x}_1,...,\vec{x}_N)$ is a microstate. Substituting the above expressions for Z_0, F_0 into (2), the free energy upper bound when the temperature $\beta > 0$, is given by

$$F_{var} \equiv F_0 + \langle H_1\rangle_0$$

$$= \begin{array}{l}-\frac{1}{\beta}N\log h^2 - \frac{1}{\beta}\log\left\{\sum_s W(s)\exp\left(\frac{\beta}{2}\sum_{i=1}^M\sum_{j\neq i}^M n_i n_j\lambda^2\log|\vec{x}_i^0 - \vec{x}_j^0|\right)\right\}\\ + \int_{D^N}\left[-\frac{1}{2}\sum_{i=1}^M\left(\sum_{j=1}^{n_i}\sum_{k\neq j}^{n_i}\lambda^2\log|\vec{x}_j' - \vec{x}_k'|\right)\right]\frac{\exp(-\beta H_0(\vec{x}))}{Z_0}d\vec{x}_1...d\vec{x}_N\end{array};$$

similarly for negative temperatures $\beta < 0$, the free energy lower bound is given by

$$F_{var}^- \equiv F_0 + \langle H_1\rangle_0$$

$$= \begin{array}{l}-\frac{1}{\beta}N\log h^2 - \frac{1}{\beta}\log\left\{\sum_s W(s)\exp\left(\frac{\beta}{2}\sum_{i=1}^M\sum_{j\neq i}^M n_i n_j\lambda^2\log|\vec{x}_i^0 - \vec{x}_j^0|\right)\right\}\\ + \int_{D^N}\left[-\frac{1}{2}\sum_{i=1}^M\left(\sum_{j=1}^{n_i}\sum_{k\neq j}^{n_i}\lambda^2\log|\vec{x}_j' - \vec{x}_k'|\right)\right]\frac{\exp(-\beta H_0(\vec{x}))}{Z_0}d\vec{x}_1...d\vec{x}_N\end{array}.$$

Whether the temperture is positive or negative, the integral in the above equation can be evaluated as follows:

$$
\sum_{s} W(s) \int\limits_{B_1^{n_1}} d^{n_1}\vec{x}' \ .... \int\limits_{B_M^{n_m}} d^{n_M}\vec{x}' \left[\begin{array}{c} -\frac{1}{2}\sum\limits_{i=1}^{M}\left(\sum\limits_{j=1}^{n_i}\sum\limits_{k\neq j}^{n_i} \lambda^2 \log|\vec{x}'_j - \vec{x}'_k| \right) \\ \exp\left(\frac{\beta}{2}\sum\limits_{i=1}^{M}\sum\limits_{j\neq i}^{M} n_i n_j \lambda^2 \log|\vec{x}_i^0 - \vec{x}_j^0| \right) \\ \times \dfrac{}{Z_0} \end{array} \right] \tag{6}
$$

$$
= -\frac{1}{2Z_0}\sum_{s} W(s) \left[\begin{array}{c} \exp\left(\frac{\beta}{2}\sum\limits_{i=1}^{M}\sum\limits_{j\neq i}^{M} n_i n_j \lambda^2 \log|\vec{x}_i^0 - \vec{x}_j^0| \right) \\ \times \sum\limits_{i=1}^{M} \int\limits_{B_i^{n_i}} d^{n_i}\vec{x}' \left(\sum\limits_{j=1}^{n_i}\sum\limits_{k\neq j}^{n_i} \lambda^2 \log|\vec{x}'_j - \vec{x}'_k| \right) \end{array} \right]
$$

$$
= -\frac{1}{2Z_0}\sum_{s} \left\{ \begin{array}{c} W(s) \prod\limits_{i\neq j}^{M} |\vec{x}_i^0 - \vec{x}_j^0|^{\frac{\beta n_i n_j \lambda^2}{2}} \\ \times \sum\limits_{i=1}^{M} \int\limits_{B_i^{n_i}} d^{n_i}\vec{x}' \left(\sum\limits_{j=1}^{n_i}\sum\limits_{k\neq j}^{n_i} \lambda^2 \log|\vec{x}'_j - \vec{x}'_k| \right) \end{array} \right\}.
$$

An application of the Mean Value Theorem implies that the integrand in each of the integrals in the last line above can be replaced as follows:

$$
\sum_{j=1}^{n_i}\sum_{k\neq j}^{n_i} \lambda^2 \log|\vec{x}'_j - \vec{x}'_k| = \frac{\lambda^2 n_i(n_i - 1)}{2} L(n_i), \tag{7}
$$

where $L(n_i)$ is a large negative constant of the order of

$$
\log|\vec{x}'_j - \vec{x}'_k| \simeq \log\frac{h}{\sqrt{n_i}}.
$$

Therefore, the term

$$
I = \sum_{i=1}^{M} \int\limits_{B_i^{n_i}} d^{n_i}\vec{x}' \left(\sum_{j=1}^{n_i}\sum_{k\neq j}^{n_i} \lambda^2 \log|\vec{x}'_j - \vec{x}'_k| \right)
$$

is given by

$$
I(n_i) = \sum_{i=1}^{M} \frac{\lambda^2 n_i(n_i - 1)}{2} L(n_i) h^{2n_i}.
$$

8

Substituting this expression in (6), we get

$$\int_{D^N} \left[-\frac{1}{2} \sum_{i=1}^{M} \left(\sum_{j=1}^{n_i} \sum_{k \neq j}^{n_i} \lambda^2 \log |\vec{x}_j' - \vec{x}_k'| \right) \right] \frac{\exp(-\beta H_0(\vec{x}))}{Z_0} d\vec{x}_1...d\vec{x}_N$$

$$\simeq \frac{1}{2Z_0} \sum_s \left\{ W(s) \left(\prod_{i \neq j}^{M} |\vec{x}_i^0 - \vec{x}_j^0|^{\frac{\beta n_i n_j \lambda^2}{2}} \right) \sum_{i=1}^{M} \frac{\lambda^2 n_i (1 - n_i)}{2} L(n_i) h^{2n_i} \right\}.$$

Thus, both F_{var}^- and F_{var} up to order $O(h)$ is given by:

$$F_{var}^- = F_{var} = -\frac{1}{\beta} \left[N \log h^2 + \log \left\{ \sum_s W(s) \exp \left(\frac{\beta}{2} \sum_{i=1}^{M} \sum_{j \neq i}^{M} n_i n_j \lambda^2 \log |\vec{x}_i^0 - \vec{x}_j^0| \right) \right\} \right] \quad (8)$$

$$- \frac{1}{4Z_0} \sum_s \left\{ W(s) \left(\prod_{i \neq j}^{M} |\vec{x}_i^0 - \vec{x}_j^0|^{\frac{\beta n_i n_j \lambda^2}{2}} \right) \sum_{i=1}^{M} \lambda^2 n_i (n_i - 1) L(n_i) h^{2n_i} \right\} + O(h) \quad .$$

2.1 Bounds

The strategy of variational mean field theories[20] at this point is to introduce some parameters γ into the expressions for F_{var} (respectively F_{var}^-) and minimize $F_{var}\{\gamma\}$ (resp. maximize $F_{var}^-\{\gamma\}$) with respect to γ. This yields the best possible approximation to the full free energy F within the class determined by the chosen reduced Hamiltonian H_0. Since

$$F = -\frac{1}{\beta} \log Z,$$

we now have the best approximation for Z. In our derivation of the mean field theory for the point vortex model, these parameters are given by the macrostate $s = \{n_1, ...n_M\}$ which are occupation numbers of the boxes B_i in the statistical coarse-graining procedure. We will now use the facts that the energy H of the system, the entropy S_0 and the partition function Z_0 are all highly focussed at the equilibrium (most probable) macrostate s^* in the mean field limit of large N. This is essentially the Landau approximation in the Ginsburg-Landau theory for phase transitions[20]. In particular, we will use the following consequence of the Landau approximation

$$P_0(s^*) = \frac{W(s^*) \left(\prod_{j \neq i}^{M} |\vec{x}_i^0 - \vec{x}_j^0|^{\frac{\beta n_i n_j}{2} \lambda^2} \right)}{\sum_s W(s) \left(\prod_{j \neq i}^{M} |\vec{x}_i^0 - \vec{x}_j^0|^{\frac{\beta n_i n_j}{2} \lambda^2} \right)} \simeq 1,$$

to analyse the expressions (8). Thus F_{var} and F_{var}^- (8) become

$$F_{var}^- = F_{var} \simeq -\frac{1}{\beta} \left[N \log h^2 + \log \left\{ W(s^*) \exp \left(\frac{\beta}{2} \sum_{i=1}^{M} \sum_{j \neq i}^{M} n_i n_j \lambda^2 \log |\vec{x}_i^0 - \vec{x}_j^0| \right) \right\} \right] \quad (9)$$

$$-\frac{1}{4} \left\{ \sum_{i=1}^{M} \lambda^2 n_i (n_i - 1) L(n_i) \right\}$$

for all $\beta \geq 0$ and for a range of values $0 > \beta > \beta^*$, where β^* is given by the theory for the existence of the mean field thermodynamic limit [4,5].

In order to minimize F_{var} and maximize F_{var}^- (9) with respect to $s = \{n_1, ..., n_m\}$ under the constraint

$$\sum_{i=1}^{M} n_i = N,$$

we augment $F_{var}(s)$ and $F_{var}^-(s)$ by adding the auxillary term with Lagrange multiplier α to obtain

$$\tilde{F}_{var}(s) = F_{var} + \alpha \left(\sum_{i=1}^{M} n_i - N \right),$$

$$\tilde{F}_{var}^-(s) = F_{var}^- + \alpha \left(\sum_{i=1}^{M} n_i - N \right),$$

Then taking the gradient $\nabla_s \tilde{F}_{var}(s) = 0$ (resp. $\nabla_s \tilde{F}_{var}^-(s) = 0$) yields

$$\frac{1}{\beta} (\log n_i + 1) + \sum_{j \neq i}^{M} \lambda^2 n_j \log |\vec{x}_i^0 - \vec{x}_j^0|$$

$$+ \lambda^2 \left[\frac{n_i(n_i - 1)}{4} L'(n_i) + \frac{2n_i - 1}{4} L(n_i) + \frac{n_i(n_i - 1)}{2} L(n_i) \right] + \alpha = 0 \quad (10)$$

for $i = 1, ..., M$. We note that $\nabla_s \tilde{F}_{var}(s)$ (resp. $\nabla_s \tilde{F}_{var}^-(s)$) differs from $\nabla_s \tilde{F}_0(s)$ by the derivatives of the term

$$\sum_{i=1}^{M} \frac{\lambda^2 n_i(n_i - 1)}{4} L(n_i), \quad (11)$$

which is associated with the self energies of each of the M boxes B_i in the coarse-graining procedure. In working with $F_{var}(s^*)$ (resp. $F_{var}^-(s)$) instead

10

of F_0, we have put back these self energy terms. Minimizing F_{var} (resp. maximizing F_{var}^- in the case of negative temperatures) yields a lower bound for F_{var} (resp. an upper bound for F_{var}^-) which provides the best approximation for the free energy F of the point vortex system[20] within the ansatz of assuming H_0 to be the approximate (mean field) Hamiltonian. Solving for n_i in (10) yields the occupation numbers

$$n_i = e^{-\alpha} \exp\left(-\beta\left(\sum_{j\neq i}^{M} \lambda^2 n_j \log|\vec{x}_i^0 - \vec{x}_j^0| + \lambda^2\left[\frac{n_i(n_i-1)}{4}L'(n_i) + \frac{2n_i^2-1}{4}L(n_i)\right]\right)\right).$$

Next we turn to the analysis of the behaviour of this expression as $N, M \to \infty$ while the vortex strength is scaled by $\frac{1}{N}$, i.e., $\lambda = \frac{1}{N}$ — it is easy to see that the following limits are valid:

$$\frac{n_i}{N} \rightarrow \xi(\vec{x}_i^0)\, d^2x, \qquad (12)$$

$$\frac{e^{-\alpha}}{Nh^2} \rightarrow d,$$

$$\frac{1}{N^2}\left[\frac{n_i(n_i-1)}{4}L'(n_i) + \frac{2n_i^2-1}{4}L(n_i)\right] \rightarrow E^1(\vec{x}_i^0),$$

$$\sum_{j\neq i}^{M} \frac{n_j}{N^2} \log|\vec{x}_i^0 - \vec{x}_j^0| \rightarrow E^0(\vec{x}_i^0).$$

Thus the mean field equations in the planar point vortex theory is given by:

$$\xi(\vec{x}) = \Delta\Psi = d\exp\left(-\beta(E^0(\vec{x}) + E^1(\vec{x}))\right), \qquad (13)$$

for all temperatures, which differs from that in the OJM theory by the self energy term $E^1(\vec{x})$.

We will now analyse the self energy expression in some detail. As we let $N, M \to \infty$ the vortex strengths have to be scaled by $\frac{1}{N}$ in order that the mean field (non-extensive) thermodynamic limit exists, as was shown by Caglioti et al[4], Kiessling[5] and Eyink and Spohn[6]. Since $N \leq M\max_i(n_i)$ and $h^2 = A/M$, we have the bound

$$L(n_i) = \frac{1}{2}\log\frac{h^2}{n_i} \geq \frac{1}{2}\log\left[h^2\left(\frac{1}{\max_i(n_i)}\right)\right]$$

11

$$= \frac{1}{2}\log A - \frac{1}{2}\log M - \frac{1}{2}\log[\max_i(n_i)]$$

$$= \frac{1}{2}\log A - \frac{1}{2}\log[M\max_i(n_i)]$$

$$\geq \frac{1}{2}\log A - \frac{1}{2}\log N,$$

which implies that for each $i = 1, ..., M$,

$$|L(n_i)| \leq \frac{1}{2}\log N - \frac{1}{2}\log A. \tag{14}$$

The self energy term in (13) scales as follows

$$
\begin{aligned}
|E^1(\vec{x})| &\leq \frac{n_i^2}{N^2}|L'(n_i)| + \frac{n_i^2}{N^2}|L(n_i)| \\
&\leq \frac{n_i^2}{N^2}|L'(n_i)| + \frac{n_i\max_i n_i}{N}\left(\frac{|L(n_i)|}{N}\right) \\
&\leq \frac{n_i^2}{N^2}|L'(n_i)| + n_i\frac{|L(n_i)|}{N},
\end{aligned}
$$

where we have used the fact that $\left(\frac{\max_i n_i}{N}\right) < 1$. The first term tends to zero as $N \to \infty$ because

$$
\begin{aligned}
\frac{n_i^2}{N^2}|L'(n_i)| &= \frac{n_i^2}{N^2}\left(\frac{1}{2n_i}\right) = \frac{n_i}{2N^2} \\
&\leq \frac{1}{2N} \to 0.
\end{aligned}
$$

By (14), the second term tends to zero as $N \to \infty$, i.e.,

$$n_i\frac{|L(n_i)|}{N} \leq \frac{n_i}{N}\left(-\frac{1}{2}\log A + \frac{1}{2}\log N\right) \to 0 ,$$

because the number $n_i \sim \frac{N}{M}$ of particles in box B_i stays about the same as the total number N of particles and the number M of equal boxes in the statistical coarse-graining procedure both tend to ∞. We have shown that

Theorem: The mean field equation (13) of the Bogoliubov-Feynman-Landau mean field theory for point vortex dynamics tends to the mean field equation (1) of the OJM theory in the mean field thermodynamic (non-extensive) limit of infinite particles $N \to \infty$, and infinite number of boxes $M \to \infty$ in the coarse-graining procedure in the definition of the approximate Hamiltonian H_0.

3 Concluding remarks

The following discussion gives a brief derivation of another formulation for F_{var} and F_{var}^-. By definition, the free energy based on the Hamiltonian H_0 is given by

$$F_0 = \langle H_0 \rangle_0 - TS_0,$$

where the expectation operator $\langle \cdot \rangle_0$ is defined by (3), and S_0 is the Gibbs entropy function based on H_0, i.e.,

$$S_0 = -k_B \sum_s P_0(s) \log P_0(s),$$

with

$$P_0(s) \equiv \frac{W(s) h^{2N} \exp(-\beta H_0(s))}{Z_0}.$$

Thus, the upper bound F_{var} (resp. lower bound F_{var}^-) for the free energy F

$$F_{var}^- = F_{var} = \langle H_0 + H_1 \rangle_0 - TS_0. \tag{15}$$

is exactly equal to the free energy based on the full Hamiltonian H, but using the probability distribution P_0.

To summarize, we have derived the OJM theory by a new method which is based on the Bogoliubov-Feynman inequality, the Gibbs entropy function and Landau's approximation, and showed that it is a mean field theory. It is remarkable that no additional conditions on the initial vorticity distributions was needed to prove this result. In the case of a two component vortex gas, our procedure gives the well-known sinh-Poisson equation[2]. The analogue of the above theorem for point vortices on a rotating sphere is presented in Lim[23]. We note that previous derivations of the OJM theory[1,2,3] are based on Boltzmann's entropy function instead of Gibbs' entropy function, which is the fore-runner of the information-theoretic entropy function.

In another paper[24], this author will return to the issue of the indeterminacy of the OJM mean field equations which was raised by Turkington [18]. This issue concerns the fact that a given continuous vorticity distribution can be represented in a number of different ways by clouds of point vortices— for example, one could use two species of vortices with equal but opposite circulations, or one could just as well choose an approximation based on three different species of vortices. The mean field equations ensuing from

13

these distinct representations of the original continuous vorticity distributions must necessarily differ; this can be demonstrated equally well within the traditional formulation and the current derivation of the OJM theory.

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